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A Hamiltonian Decomposition of K^*_{2m}, $2m \geqslant 8$

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It is shown that K^*_{2m}, $2m \geqslant 8$, can be decomposed into Hamiltonian circuits. A direct construction utilizing difference methods is given for $2m \equiv 0 \pmod 4$. The case $2m \equiv 2 \pmod 4$ is handled inductively by means of a construction which shows that K^*_{4m-2} admits such a decomposition if K^*_{2m} does.

In this paper a constructive proof utilizing difference methods will be given of the following conjectures due to Bermond and Faber:

Conjecture 1. For $2m \geqslant 8$, the edges of the complete directed graph on $2m-1$ vertices K^*_{2m-1} can be partitioned into $2m-1$ directed Hamiltonian paths $P_1, P_2, \ldots, P_{2m-1}$.

Conjecture 2. For $2m \geqslant 8$, the edges of the complete directed graph on $2m$ vertices K^*_{2m} can be partitioned into $2m-1$ directed Hamiltonian circuits $C_1, C_2, \ldots, C_{2m-1}$.

These conjectures are made in the article of Bermond and Faber [2], "Decomposition of the Complete Directed Graph into k-Circuits," where they are shown to be equivalent. They note that the conjecture is false for $2m = 4$ and 6, while the existence of sequenceable groups of orders 21, 39, 55 and 57 give solutions to (2) for $2m = 22$, 40, 56 and 58. The authors of this article have also verified (2) for $8 \leqslant 2m \leqslant 18$ by computer. Keedwell has obtained a sequencing of the non-abelian group of order 27 in [3], giving a solution for $2m = 28$.

The fact that the edges of K_{2m+1}, $m \geqslant 1$, can be decomposed into Hamiltonian cycles was known to Kirkman [1]. A type (2) decomposition for K^*_{2m+1}, $m \geqslant 1$, is an immediate corollary of this result. One such decomposition, which will be used in the proof of Proposition 2, will now be described.

A Latin square L of order n is called *row complete* if for all ordered pairs (u, v) of distinct symbols there exist integers r and c, $1 \leqslant r \leqslant n$, $1 \leqslant c \leqslant n-1$, such that $u = L(r, c)$ and $v = L(r, c+1)$. In [4], Williams

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published a construction for row complete latin squares of order $n = 2m$, $m \geqslant 1$, on the symbol set $\{0, 1, \ldots, n-1\}$. It is given by

$$\begin{aligned} L(1, c) &= c/2 \ (\mathrm{mod}\ n), && \text{if } c \text{ is even} \\ &-((c-1)/2) \ (\mathrm{mod}\ n), && \text{if } c \text{ is odd}, \\ L(r, c) &= r - 1 + L(1, c) \ (\mathrm{mod}\ n), && 1 \leqslant c \leqslant n, 2 \leqslant r \leqslant n. \end{aligned}$$

The verification that L is a row complete latin square is easy and is omitted. If we regard the symbols $\{0, 1, \ldots, n-1\}$ as vertices and pairs of adjacent symbols in a row as edges, this construction gives a decomposition of type (1) for K^*_{2m}, $m \geqslant 1$. By adjoining the new symbol ψ in column $n+1$ of each row of a row complete latin square on $n = 2m$ symbols, and forming cycles out of each row, we have constructed a decomposition of type (2) for K^*_{2m+1}, $m \geqslant 1$.

A direct construction of the decomposition of K^*_{2m} when $2m \equiv 0 \pmod 4$ will be given in Proposition 2. The method is suggested by the following construction of a type (1) decomposition for $2m = 10$. We start with a type (2) decomposition of K^*_9, given in Fig. 1 by a row complete latin square of order

ψ	0	1	7	2	6	3	5	4
ψ	1	2	0	3	7	4	6	5
ψ	2	3	1	4	0	5	7	6
ψ	3	4	2	5	1	6	0	7
ψ	4	5	3	6	2	7	1	0
ψ	5	6	4	7	3	0	2	1
ψ	6	7	5	0	4	1	3	2
ψ	7	0	6	1	5	2	4	3

Rows are regarded as circuits.

FIG. 1. Type (2) decomposition of K^*_9.

eight with the symbol ψ adjoined in the first column of every row. As before, the rows are regarded as being cyclically linked. The set α of underlined edges has the property that each circuit contains exactly one of its elements, and that it is the edge set of a Hamiltonian path in K^*_9. If we delete the underlined edge from each of the existing circuits the eight resulting Hamiltonian paths together with the path whose edge set is α form a type (1) decomposition of K^*_9.

PROPOSITION 2. *For $2m \equiv 0 \pmod 4$, $2m \geqslant 8$, Conjecture 1 is true.*

Proof. In this and the proofs that follow, $V(G)$ and $E(G)$ will denote the vertex and edge sets, respectively, of a graph G.

Column	1	2	3		$2k-1$	$2k$			$4k-1$	$4k$	$4k+1$	$4k+2$
ψ	0	1	$4k+1$	. . .	$3k+3$	k		. . .	$2k+3$	$2k$	$2k+2$	$2k+1$
ψ	1	2	0	. . .	$3k+4$	$k+1$		. . .	$2k+4$	$2k+1$	$2k+3$	$2k+2$
.	.	.	.			.	k edges		.	.	.	.
.	.	.	.			.	underlined here		.	.	.	.
.	.	.	.		1	$2k$			.	.	.	.
.	.	.	.		2	$2k+1$			.	.	$3k+3$	$3k+2$
.	.	.	.		3	$2k+2$			.	.	.	.
.	.	.	.			.	k edges		.	.	.	.
.	.	.	.			.	underlined here		.	.	.	.
.	.	.	.		$k+2$	$3k+1$			.	.	.	.
ψ	$2k+2$	$2k+3$	$2k+1$	. . .	$k+3$	$3k+2$			.	.	.	.
.	.	.	.		$k+4$	$3k+3$			.	.	.	.
.	.	.	.			.	$k-1$ edges		.	.	.	.
.	.	.	.			.	underlined here		.	.	.	.
.	.	.	.		$2k+2$	$4k+1$			.	.	.	.
.	.	.	.		$2k+3$	0		. . .	$k+3$	k	.	.
.	.	.	.		$2k+4$	1			.	.	.	.
.	.	.	.			.	$k-1$ edges		.	.	.	.
.	.	.	.			.	underlined here		.	.	.	.
ψ	$4k+1$	0	$4k$	. . .	$3k+2$	$k-1$		. . .	$2k+2$	$2k-1$	$2k+1$	$2k$

FIG. 2. A portion of the Williams row complete latin square of order $n = 4k+2$, $k \geqslant 1$. The rows are regarded as circuits.

By the previous discussion, we have a decomposition of type (2) for K^*_{2m-1}, that is, a partition of the edges into $2m-2$ directed Hamiltonian circuits $C_1, C_2, \ldots, C_{2m-2}$. This construction, using a row complete latin square of order $n = 2m-2 = 4k+2$, $k \geqslant 1$, is illustrated in Fig. 2.

The proof consists of exhibiting a set of edges $\alpha = \{e_i \mid 1 \leqslant i \leqslant 2m-2\}$ in this construction (which are underlined in Fig. 2) such that

$$e_i \in E(C_i), \qquad 1 \leqslant i \leqslant 2m-2 \tag{3}$$

and

$$\alpha \text{ is the edge set of a directed Hamiltonian path in } K^*_{2m-1}. \tag{4}$$

As before, by deleting e_i from C_i for each i, and setting $\hat{C}_i = C_i - e_i$, we obtain a partition $E(\hat{C}_1) \cup E(\hat{C}_2) \cup \cdots \cup E(\hat{C}_{2m-2}) \cup \alpha$ of the edges of K^*_{2m-1} into $2m-1$ directed Hamiltonian paths.

Inspection of Fig. 2 shows immediately that α satisfies property (3).

One way to see that (4) holds for α follows. To any edge e in $E(K^*_{2m-1})$ not incident with the vertex ψ we can associate its difference $d(e) =$ head$(e) -$ tail(e) (mod $n = 4k+2$). Let $\beta = \{e \in E(K^*_{2m-1}) \mid d(e) = 2k-1 \pmod{4k+2}\}$; these are the edges formed by columns $2k-1$ and $2k$ in Fig. 2. Since $2k-1$ and $4k+2$ are relatively prime, β is the edge set of a directed $(4k+2)$-gon R in K^*_{2m-1}, with $V(R) = V(K^*_{2m-1}) - \{\psi\}$. The exact form of R depends upon the congruence class of k (mod 4), and can be determined by solving congruences of the type $a + x(2k-1) \equiv b \pmod{4k+2}$ where x gives the number of edges of R in the path $(a = a_0, a_1, a_2, \ldots, a_{x-1}, a_x = b)$. These computations are facilitated by the relations given in the following table.

TABLE I

Useful Relations mod $4k+2$

Value of k (mod 4)	$(2k-1)^{-1}$	k^2
0	$-k-1$	$-(k/2)$
1	k	$(k+1)/2$
2	$-k-1$	$-(k/2)+2k+1$
3	k	$(k+1)/2+2k+1$

As an example, if $k \equiv 1 \pmod 4$ then by solving $0 + x_1(2k-1) \equiv 3k+3 \pmod{4k+2}$ using the relations given above, we see that there are $x_1 = (k-1)/2$ edges of R in the directed path from vertex 0 to vertex $3k+3$.

Let $\beta = \{(3k+3, k), (2, 2k+1), (k+3, 3k+2), (2k+3, 0)\}$ denote the four edges of R not in α and set $\gamma = \alpha \backslash E(R)$. The two different orders of occurrence of the edges of β in R as a function of k (mod 4) are shown in Fig. 3a and b; the edges in β are dashed.

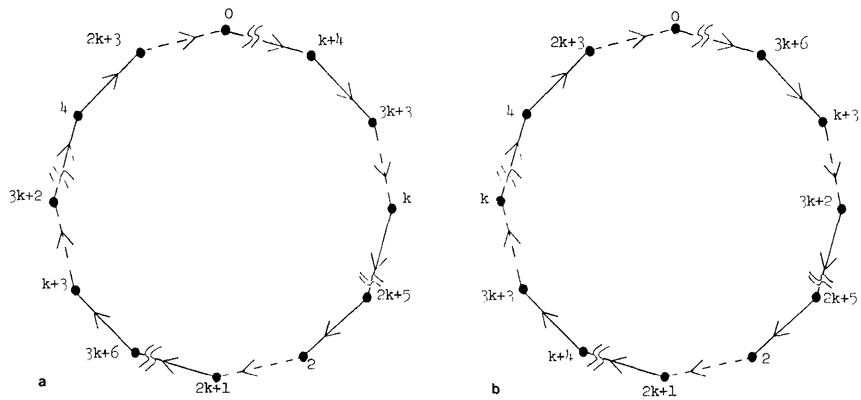

FIG. 3. (a) R when $k \equiv 1$ or 2 (mod 4). The four edges of R not in α are dashed. (b) R when $k \equiv 0$ or 3 (mod 4). The four edges of R not in α are dashed.

Figure 4 shows R with the edges in β deleted and the edges in γ added, that is, the simple open directed path P with edge set α. It is clear that P is Hamiltonian since $|E(P)| = |E(R)| - 4 + 4 = 4k + 2$ and $V(P) = V(R) \cup \{\psi\} = V(K^*_{2m-1})$. Thus (4) holds and the proof of Proposition 2 is complete.

Let $K_{S,\bar{S}}$ be the complete bipartite undirected graph with vertex set $S \cup \bar{S}$, where we take $\bar{S} = (\bar{x} \mid x \in S\}$. The cocktail party graph $C_{S,\bar{S}}$ is defined by $V(C_{S,\bar{S}}) = V(K_{S,\bar{S}})$ and $E(C_{S,\bar{S}}) = E(K_{S,\bar{S}}) - M$, where M is the edge set of a perfect matching of $K_{S,\bar{S}}$, say $M = \{\{x, \bar{x}\} \mid x \in S\}$. $K^*_{S,\bar{S}}$ and $C^*_{S,\bar{S}}$ will denote

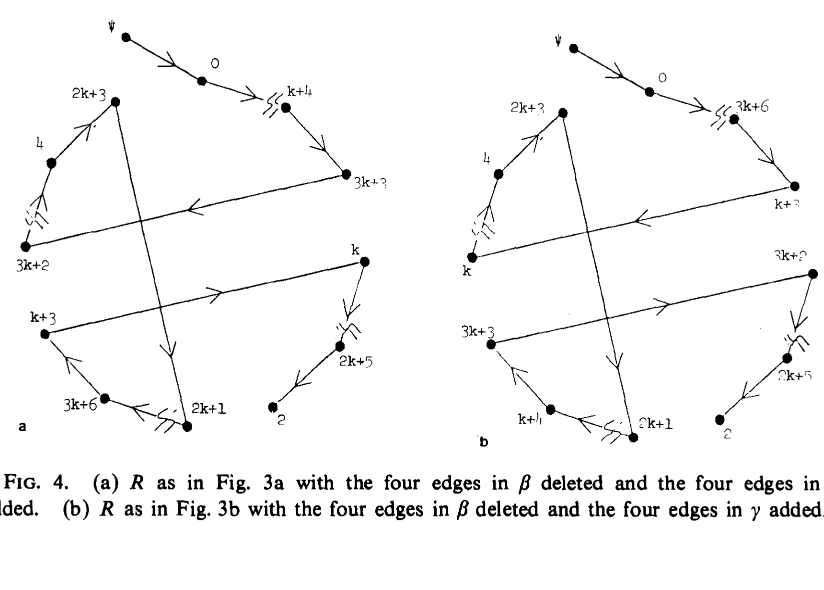

FIG. 4. (a) R as in Fig. 3a with the four edges in β deleted and the four edges in γ added. (b) R as in Fig. 3b with the four edges in β deleted and the four edges in γ added.

the complete directed and directed cocktail party graphs with vertex set $S \cup \bar{S}$; if $|S| = p$, we will also use the notation $K^*_{p,p}$ and $C^*_{p,p}$ for these graphs.

PROPOSITION 3. *For $m \geqslant 2$, the edges of $C^*_{2m-1,2m-1}$ can be partitioned into $2m-2$ directed Hamiltonian cycles.*

Proof. This proposition is a consequence of the well-known result that if $n \geqslant 3$ is odd, $K_{n,n}$ can be decomposed into hamiltonian cycles and a perfect matching.

PROPOSITION 4. *If K^*_{2m} admits decomposition* (2), *then so does K^*_{4m-2}.*

Proof. Let $V(K^*_{4m-2}) = S \cup \bar{S}$, where we take $S = \{0, 1, ..., 2m-3, \infty\}$, with $\bar{S}$ and $C^*_{2m-1,2m-1}$ as above. By Proposition 3 all we need do is partition $E(K^*_{4m-2}) - E(C^*_{2m-1,2m-1})$ into $2m-1$ directed Hamiltonian cycles.

By the hypothesis and the equivalence of (1) and (2) we have a decomposition of K^*_{2m-1} into $2m-1$ directed Hamiltonian paths $P_1, P_2, ..., P_{2m-1}$. Set $V(K^*_{2m-1}) = S$. We can define $2m-1$ directed Hamiltonian paths $\bar{P}_1, \bar{P}_2, ..., \bar{P}_{2m-1}$ on the vertex set $\bar{S}$ by

$$V(\bar{P}_i) = \{\bar{z} : z \in V(P_i)\},$$
$$E(\bar{P}_i) = \{(\bar{x}, \bar{y}) : (y, x) \in E(P_i)\}, \qquad 1 \leqslant i \leqslant 2m-1.$$

We can link P_i and $\bar{P}_i$ to form $2m-1$ directed Hamiltonian circuits H_i of $E(K^*_{4m-2}) - E(C^*_{2m-1,2m-1})$ by adding the edges $(T(P_i), I(\bar{P}_i))$ and $(T(\bar{P}_i), I(P_i))$, $1 \leqslant i \leqslant 2m-1$. Note that the added edges are the set $M^* = \{(x, \bar{x}), (\bar{x}, x) : x \in S\}$.

It is clear that $E(H_i) \cap E(H_j) = \varnothing$ for $i \neq j$, $1 \leqslant i, j \leqslant 2m-1$, since $E(P_i) \cap E(P_j) = E(\bar{P}_i) \cap E(\bar{P}_j) = \varnothing$ and $\{T(P_i) : 1 \leqslant i \leqslant 2m-1\} = \{I(P_i) : 1 \leqslant i \leqslant 2m-1\} = S$, as was established in the introductory remarks.

THEOREM. *For $2m \geqslant 8$, the edges of the complete directed graph on $2m$ vertices K^*_{2m} can be partitioned into $2m-1$ directed Hamiltonian circuits $C_1, C_2, ..., C_{2m-1}$.*

Proof. For $m = 4$ the theorem is true by Proposition 2. The solution for $m = 5$ has already been given in the introductory remarks. We now proceed by induction on m, $m \geqslant 5$. If $2m \equiv 0 \pmod 4$ we are done by Proposition 2. Suppose $2m \equiv 2 \pmod 4$, say $2m = 4l - 2$ where $l \geqslant 4$. Since $2l < 2m$, the theorem is true for K^*_{2l} by either the remarks above or the inductive hypothesis. Proposition 4 then gives the result for $K^*_{4l-2} = K^*_{2m}$.

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